PHILOSOPHICAL TRANSACTIONS.

XIX.—*Consideration of the objections raised against the geometrical representation of the square roots of negative quantities. By the Rev.* JOHN WARREN, *M.A. of Jesus College, Cambridge. Communicated by* THOMAS YOUNG, *M.D. Foreign Secretary to the Royal Society.*

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SOME years ago my attention was drawn to those algebraic quantities, which are commonly called impossible roots or imaginary quantities: it appeared extraordinary, that mathematicians should be able by means of these quantities to pursue their investigations, both in pure and mixed mathematics, and to arrive at results which agree with the results obtained by other independent processes; and yet that the real nature of these quantities should be entirely unknown, and even their real existence denied. One thing was evident respecting them; that they were quantities capable of undergoing algebraic operations analogous to the operations performed on what are called possible quantities, and of producing correct results: thus it was manifest, that the operations of algebra were more comprehensive than the definitions and fundamental principles; that is, that they extended to a class of quantities, viz. those commonly called impossible roots, to which the definitions and fundamental principles were inapplicable. It seemed probable, therefore, that there was a deficiency in the definitions and fundamental principles of algebra; and that other definitions and fundamental principles might be discovered of a more comprehensive nature, which would extend to every class of quantities to which the operations of algebra were applicable; that is, both to possible and impossible quantities, as they are called. I was induced therefore to examine into the nature of algebraic operations, with a view, if possible, of arriving at these general definitions and fundamental principles: and I found, that, by considering algebra merely as applied to geometry, such principles and definitions might be obtained. The fundamental principles and definitions which I arrived at were these: that all straight lines drawn in a given plane from a given point, in any direction whatever, are capable of being algebraically represented, both in length and direction; that the addition of such lines (when estimated both in length and direction) must be performed in the same manner as composition of motion in dynamics; and that four such lines are proportionals, both in length and direction, when they are proportionals in length, and the fourth is inclined to the third at the same angle that the second is to the first. From these principles I deduced, that, if a line drawn in any given direction be assumed as a positive quantity, and consequently its opposite, a negative quantity, a line drawn at right angles to the positive or negative direction will be the square root of a negative quantity, and a line drawn in an oblique direction will be the sum of two quantities, the one either positive or negative, and the other, the square root of a negative quantity.

This may be illustrated by the following examples:

(1) Let it be required to find the length and direction of $\sqrt{-1}$;

First to find the direction of $\sqrt{-1}$,

$\sqrt{-1}$ is evidently a mean proportional between $+1$, and -1;

Now by the definition of proportion, if 4 lines be proportionals, the fourth is inclined to the third at the same angle that the second is to the first,

$\therefore$ if three lines be proportionals, the third is inclined to the second at the same angle that the second is to the first,

$\therefore$ a mean proportional between any two lines must lie in such a direction as to bisect the angle at which those lines are inclined to each other,

$\therefore \sqrt{-1}$ bisects the angle at which -1 is inclined to $+1$;

But -1 is inclined to $+1$ at $180°$,

$\therefore \sqrt{-1}$ is inclined to $+1$ at $90°$;

Next to find the length of $\sqrt{-1}$;

Since $\sqrt{-1}$ is a mean proportional between $+1$ and -1, and $+1$ and -1 are equal in length, $\sqrt{-1}$ is equal in length either to $+1$ or -1;

$\therefore \sqrt{-1}$ is a line equal in length to $+1$, and drawn at right angles to $+1$.

(2) Hence, if a be a positive quantity, a line equal in length to a and drawn at right angles to $+1$ will be equal to $a\sqrt{-1}$.

(3) Let it be required to find the length and direction of $\sqrt[4]{-1}$; $\sqrt[4]{-1} = \sqrt{\sqrt{-1}}$, $\therefore \sqrt[4]{-1}$ is a mean proportional between $+1$ and $\sqrt{-1}$,

$\therefore \sqrt[4]{-1}$ is equal in length to $+1$;

Also it has been proved that $\sqrt{-1}$ is inclined to $+1$ at 90°,

$\therefore \sqrt[4]{-1}$ is inclined to $+1$ at 45°;

$\therefore \sqrt[4]{-1}$ is a line equal in length to $+1$, and inclined to $+1$ at 45°.

(4) As $\sqrt[4]{-1}$ is a line drawn in an oblique direction, let it be required to find an expression for it, considered as the sum of two quantities, the one either positive or negative, and the other the square root of a negative quantity.

Since $\sqrt[4]{-1}$ is equal in length to $+1$ and is inclined to $+1$ at 45°, and addition is performed in the same manner as composition of motion,

$$\sqrt[4]{-1} = \cos 45^\circ + \sin 45^\circ \sqrt{-1}$$
$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \cdot \sqrt{-1}.$$

(5) To show that $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\sqrt{-1}$ is a true value of $\sqrt[4]{-1}$ according to common algebra;

Let $\sqrt[4]{-1} = x$,

then $x^4 + 1 = 0$,

an equation, one of whose roots is $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\sqrt{-1}$,

$$\therefore \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\sqrt{-1} \text{ is a true value of } \sqrt[4]{-1}.$$

In like manner other examples might be given, but these will suffice to illustrate the definition; and for further information I must refer to a treatise which I published on this subject in April 1828.

Since the publication of that work, several objections have been made to the geometrical representation of the square roots of negative quantities. First, that impossible roots are merely signs of impossibility; that, if, in the solution of any question, we arrive at an equation, all whose roots are impossible, the only conclusion to be drawn, is, that the question involves an impossibility; and therefore it is absurd to suppose that the square roots of negative quantities can have any real existence. A second objection is, that there is no necessary connexion between algebra and geometry, and therefore that it is improper to introduce geometric considerations into questions purely algebraic; and that the geometric representation, if any exists, can only be analogical, and not a true algebraic representation of the roots. A third objection is, that this geometric representation, even if it be a correct representation of the roots, is merely a matter of curiosity, and can be of no use to mathematicians.—The object of this paper is to answer these objections.

To the first objection, that impossible roots are merely signs of impossibility, it may be replied, that, though they are so in some questions they are not necessarily so in all, and that in this respect they resemble fractional and negative roots. It is obvious, that in all equations derived from suppositions, which involve an impossibility or absurdity, the impossibility or absurdity will show itself in the result of the operation: and this may appear as well by a fractional root or by a negative root, as by a root commonly called impossible: thus, if we have a question, which from its nature does not admit of a fractional answer, and in resolving this question we arrive at an equation, which only admits of fractional roots, these fractional roots are in this case a proof, that the question involves an impossibility. Also if a question does not admit of a negative answer, and in resolving it we arrive at an equation which only admits of negative roots, in this case also we conclude that the question involves an impossibility. So, in like manner, if in resolving a question which does not admit of what are now commonly called impossible roots as answers, we arrive at an equation, all of whose roots are impossible, we must conclude, that the question involves an impossibility; and we have no greater reason for inferring from the last case, that what are called impossible roots have no real existence, than we have for inferring from the two former cases, that fractional or negative quantities have no real existence. This will be rendered clearer by an example: Let a body revolve in a circle by the action of a centripetal force, which varies inversely as the nth power of the distance; and let it be required to find the height from which a body must fall to the circle to acquire the velocity of the body revolving in the circle. In this example, if n be greater than 3, the velocity of the body revolving in the circle is greater than the velocity which can be acquired by falling from any distance however great; therefore the question involves an impossibility: therefore, if we obtain an equation for determining the height from which a body must fall to acquire a velocity equal to the velocity in the circle, the equation must, when n is greater than 3, show, in some way, that the question involves an impossibility.

Let r be the radius of the circle, and x the height (measuring from the centre) from which the body must fall to acquire the velocity in the circle; then we obtain the following equation.

$$\frac{1}{r^{n-1}} = \frac{2}{n-1} \cdot \left\{ \frac{1}{r^{n-1}} - \frac{1}{x^{n-1}} \right\}.$$

From which we deduce $x^{n-1} - \frac{2}{3-n} \cdot r^{n-1} = 0$.

Now, by the nature of the question, x must be positive; therefore whenever the above equation has no positive root, the question must involve an impossibility.

Let $n = 5$, then $x^4 + r^4 = 0$, an equation, all whose roots are what are called impossible roots; therefore, since the equation has no positive root, the question involves an impossibility.

Next, let $n = 6$, then $x^5 + \frac{2}{3} r^5 = 0$, an equation, one of whose roots is negative, and the other four what are called impossible roots; therefore in this case also, since the equation has no positive root, the question involves an impossibility.

Therefore a negative root may be a sign of impossibility, as well as what is called an impossible root.

In like manner other examples might be given, from which it would appear, that, in some cases, fractional roots may also be signs of impossibility.

Therefore we have no stronger reasons a priori to determine, that, what are called impossible roots, have no real existence, because, in some cases, they are signs of impossibility, than we have to determine that fractional or negative roots have no real existence, because, in some cases, they also are signs of impossibility.

To the second objection, viz. that there is no necessary connexion between algebra and geometry, it may be answered, that there is a connexion between what are called impossible roots and the series for the circumference of the circle, which connexion may be proved on principles purely algebraic, without the intervention of any geometric considerations.

This will appear from the expansion of 1^x;

One of the values of 1^x is 1,

This value is not a function of x,

But 1^x has other values, which are functions of x;

For example, let $x = \frac{1}{3}$, and let $1^{\frac{1}{3}} = y$,

$$\text{then } y^3 - 1 = 0,$$

an equation, whose roots are $1, \frac{-1+\sqrt{-3}}{2}, \frac{-1-\sqrt{-3}}{2}$;

Next, let $x = \frac{1}{4}$, and let $1^{\frac{1}{4}} = y$,

$$\text{then } y^4 - 1 = 0,$$

an equation whose roots are $1, -1, +\sqrt{-1}, -\sqrt{-1}$;

In like manner it will appear, if other values be given to x, that 1^x will have values which are dependent upon the values of x;

$$\text{that is, } 1^x \text{ is a function of } x;$$

$\therefore$ 1^x may be expressed in the form $A + Bx + Cx^2 +$ &c.
where A, B, C, &c. are constant quantities independent of the value of x;

First, to find the value of A, Let $x = 0$, then $1^0 = A$,

But $1^0 = 1$, $\therefore A = 1$; $\therefore 1^x = 1 + Bx + Cx^2 +$ &c.;

Next, to find the law of the series, $1^x \cdot 1^y = 1^{x+y}$,

$\therefore$ the series is of the form, $1 + Bx + \frac{B^2 x^2}{1.2} + \frac{B^3 x^2}{1.2.3} +$ &c.;

Next, to find the value of B, $1^{nx} = 1 + Bnx + \frac{B^2 n^2 x^2}{1.2} +$ &c.;

Let $\frac{1^n - 1}{1^n + 1} = m$, then $1^n = \frac{1+m}{1-m}$,

$$\therefore 1^{nx} = \frac{(1+m)^x}{(1-m)^x} = (1+m)^x \cdot (1-m)^{-x};$$

Let $M = m - \frac{1}{2}m^2 + \frac{1}{3}m^3 - \frac{1}{4}m^4 +$ &c.

$M' = -m - \frac{1}{2}m^2 - \frac{1}{3}m^3 - \frac{1}{4}m^4 -$ &c.

Then $(1+m)^x = 1 + Mx + \frac{M^2 x^2}{1.2} +$ &c.

$$(1-m)^{-x} = 1 - M'x + \frac{M'^2 x^2}{1.2} - \text{&c.}$$

$$\therefore 1^{nx} = \left(1 + Mx + \frac{M^2 x^2}{1.2} + \text{&c.}\right) . \left(1 - M'x + \frac{M'^2 x^2}{1.2} - \text{&c.}\right)$$

$$= 1 + (M - M') + x\frac{(M - M')^2 x^2}{1.2} + \&c.$$

$$\therefore\ 1 + Bnx + \frac{B^2 n^2 x^2}{1.2} + \&c. \ = 1 + (M - M')\,x + \frac{(M - M')^2 x^2}{1.2} + \&c.$$

$$\therefore\ Bn = M - M' = \left\{\begin{array}{l} m - \tfrac{1}{2}m^2 + \tfrac{1}{3}m^3 - \&c. \\ + m + \tfrac{1}{2}m^2 + \tfrac{1}{3}m^3 + \&c. \end{array}\right\}$$

$$= 2\left\{m + \frac{m^3}{3} + \frac{m^5}{5} + \&c.\right\};$$

Let $n = \frac{1}{6}$; then $m = \dfrac{1^{\frac{1}{6}} - 1}{1^{\frac{1}{6}} + 1}$;

Let $1^{\frac{1}{6}} = y$; then $y^6 - 1 = 0$,

an equation, one of whose roots is $\dfrac{1 + \sqrt{-3}}{2}$;

$\therefore$ substituting this value for $1^{\frac{1}{6}}$,

$$m = \frac{\dfrac{1 + \sqrt{-3}}{2} - 1}{\dfrac{1 + \sqrt{-3}}{2} + 1} = \frac{-1 + \sqrt{-3}}{3 + \sqrt{-3}} = \frac{(\sqrt{3} + \sqrt{-1}).\sqrt{-1}}{3 + \sqrt{-3}}$$

$$= \frac{(\sqrt{3} + \sqrt{-1}).\sqrt{-1}}{(\sqrt{3} + \sqrt{-1}).\sqrt{3}} = \frac{\sqrt{-1}}{\sqrt{3}};$$

$$\therefore\ B.\tfrac{1}{6} = 2\left\{\frac{\sqrt{-1}}{\sqrt{3}} + \tfrac{1}{3}\left(\frac{\sqrt{-1}}{\sqrt{3}}\right)^3 + \tfrac{1}{5}.\left(\frac{\sqrt{-1}}{\sqrt{3}}\right)^5 + \&c.\right\},$$

$$\therefore\ B = 12\left\{\frac{1}{\sqrt{3}} - \tfrac{1}{3}\left(\frac{1}{\sqrt{3}}\right)^3 + \tfrac{1}{5}.\left(\frac{1}{\sqrt{3}}\right)^5 - \&c.\right\}.\sqrt{-1};$$

Now $12\left\{\dfrac{1}{\sqrt{3}} - \tfrac{1}{3}\left(\dfrac{1}{\sqrt{3}}\right)^3 + \tfrac{1}{5}\left(\dfrac{1}{\sqrt{3}}\right)^5 - \&c.\right\}$ is a series, which expresses the value of the circumference of a circle, whose radius is unity;

Let this series $= c$, then $B = c\sqrt{-1}$,

$$\therefore\ 1^x = 1 + cx\sqrt{-1} - \frac{c^2 x^2}{1.2} - \frac{c^3 x^3}{1.2.3}\sqrt{-1} + \frac{c^4 x^4}{1.2.3.4} + \&c.;$$

$\therefore$ We have by means of mere algebraical operations, without the introduction of any geometrical considerations, expanded 1^x in a series, which involves c, the circumference of a circle whose radius is unity.

And this series was obtained by substituting for 1^n one of its impossible values as they are called.

$\therefore$ There is a connexion between what are called impossible roots and the circumference of the circle.

But by examining the series more accurately, we may find a greater connexion between the series and the properties of the circle

$$\text{For } 1 - \frac{c^2 x^2}{1.2} + \frac{c^4 x^4}{1.2.3.4} - \&c. = \cos c x,$$

$$\text{and } c x - \frac{c^3 x^3}{1.2.3} + \frac{c^5 x^5}{1.2.3.4.5} - \&c. = \sin c x,$$

$$\therefore 1^x = \cos c x + \sin c x \,.\, \sqrt{-1}.$$

Now lest there should be any error in the method of expanding 1^x as given above, let us try whether we can verify the last equation by some independent process;

$$\text{Let } x = \frac{p}{q}, \text{ where } p \text{ and } q \text{ are whole numbers,}$$

$$\text{and let } 1^{\frac{1}{q}} = y \text{ then } y^q = 1,$$

an equation, one of whose roots is $\cos \frac{c}{q} + \sin \frac{c}{p} \sqrt{-1}$, where c is the circumference of a circle, whose radius is unity,

$$\therefore 1^{\frac{1}{q}} = \cos \frac{c}{q} + \sin \frac{c}{q} \cdot \sqrt{-1},$$

$$\therefore 1^{\frac{p}{q}} = \left(\cos \frac{c}{q} + \sin \frac{c}{q} \cdot \sqrt{-1}\right)^p = \cos \frac{p}{q} c + \sin \frac{p}{q} c \cdot \sqrt{-1},$$

$$\text{or } 1^x = \cos c x + \sin c x \,.\, \sqrt{-1},$$

the same equation as that obtained above by the expansion of 1^x.

From what has just been proved, it appears that there is a connexion between the properties of the circle and the quantities commonly called impossible roots, that is between geometry and algebra; therefore it is so far from being improper to introduce geometric considerations into questions purely algebraic, that it is to geometry we must look (and to geometry alone as far as we know at present), if we expect to arrive at a true theory respecting the square roots of negative quantities.

It may be proper to observe here, that the object of the above investigation is not to expand all the values of 1^x in a series, but merely to show that one of them can be so expanded without the intervention of any geometric considerations, and to point out that the series obtained will involve c the circumference of the circle, and thus to prove that there is a connexion between algebra and geometry; if the object had been to obtain all the values of 1^x in a series, it would have been more convenient to have introduced geometrical considerations as in my treatise.

To the second part of this objection, viz. that the geometric representation can only be analogical and not a true algebraic representation of the roots; it may be replied, that the geometric representation of the square roots of negative quantities rests on the same foundation as the geometric representation, or any other representation of the negative quantities themselves. A negative quantity arithmetically considered is a mere absurdity, being the difference which arises from subtracting a greater quantity from a less; but algebraists having found that operations might more easily be performed by considering negative quantities in the abstract, endeavoured to establish their real existence, and with this view they made the following hypotheses; that, if a line drawn in one direction be represented by a positive quantity, a line drawn in the opposite direction will be represented by a negative quantity; that the sum of a positive and a negative quantity is to be found by subtracting the less from the greater, and prefixing the sign of the greater; that subtraction is to be performed by changing the sign of the quantity to be subtracted, and proceeding as in addition; that the product of a positive and a negative quantity is negative, and the product of two negative quantities, positive; and having made these hypotheses, they proved, by examining into the nature of algebraic operations, that the results arrived at by means of these hypotheses must be correct; therefore they concluded that these were true hypotheses; and their truth being established, they were admitted as fundamental principles of algebra: and in the same way other true hypotheses were established relative to the representation of negative quantities: such as; if time to come be represented by a positive quantity, time past will be represented by a negative quantity, &c. I call these algebraic principles, hypotheses; for though most algebraists have considered them as propositions, and have endeavoured to establish their truth by direct demonstration, yet their reasoning is unsatisfactory, for they always treat of negative quantities as quantities to be subtracted, therefore their proofs are only applicable to the difference of two positive quantities, and not to negative quantities abstractedly considered. These fundamental principles must therefore be looked upon as hypotheses introduced into algebra in order to give to negative quantities a representation and a real existence. And in like manner, in order to arrive at the representation of the square roots of negative quantities, I have made the following hypotheses: that all straight lines drawn in a given plane from a given point in any direction whatever, may be algebraically represented both in length and direction: that addition is performed in the same manner as composition of motion in dynamics; that four straight lines are proportionals, both in length and direction, when they are proportionals in length, and the fourth is inclined to the third at the same angle at which the second is inclined to the first: and I have by means of these hypotheses as a foundation, established all the common rules for performing algebraic operations, and thus have proved, that the results arrived at by means of these hypotheses must be correct: therefore I conclude, that these are true hypotheses, and true in the same sense, that the hypotheses made by algebraists respecting the representation of negative quantities are true. In fact, if there be a question, whether negative quantities can or cannot be represented geometrically; the only way in which such a question can be solved, is by making certain hypotheses with respect to their geometric representation, and then showing that the results arrived at from these hypotheses must be correct: and in like manner if there be a question whether those quantities commonly called impossible can be geometrically represented, the question must be solved in the same way; viz. by making certain hypotheses respecting them, and showing that the results arrived at by means of these hypotheses must be correct. In this point of view, the definitions and fundamental principles which I have laid down in my treatise must be considered as mere hypotheses; and mathematicians will be satisfied of their correctness when they see that the results agree in every respect with the results obtained by other independent processes.

To the third objection, viz. that the geometric representation of the square roots of negative quantities can be of no use to mathematicians, it will not be necessary to say much in reply.

In the works which have lately been written, either on pure or mixed mathematics, we may observe that great use is made of impossible roots; and we may fairly conclude that if these quantities are of so great service to mathematicians, even while they are ignorant of their real nature, they will be of much greater service when the true theory respecting them is known; we may reasonably expect, that our knowledge of algebra will be increased when the nature of impossible roots is understood in the same manner as that of possible roots; these are the general advantages which we shall derive from the geometric representation of the square roots of negative quantities: but there is one particular advantage, and that, one of the greatest importance, which arises from the definition of addition; addition is performed in the same manner as composition of motion in dynamics, therefore any question in dynamics where the motion of the bodies is confined to one plane, becomes a mere question of algebra, the laws of motion being contained in the definitions of algebra.

Before I conclude this paper, it will be proper to take notice of two works which have appeared on this subject; the first a paper in the Philosophical Transactions, for the year 1806, p. 23: intitled "Mémoire sur les Quantités Imaginaires, par M. BUÉE;" the second, a work intitled "La vraie Théorie des Quantités Négatives et des Quantités prétendues Imaginaires, par C. V. MOUREY, Paris, 1828." I was not aware of the existence of M. BUÉE's paper till November 1827, when my treatise was in the press: at that time I read his paper, and also the article upon it in the Edinburgh Review of July 1808. M. BUÉE begins with stating that the negative sign has two different significations in algebra; viz. that if algebra be considered as a universal arithmetic, the negative sign is a sign of subtraction, but that if algebra be considered as a mathematical language, the negative sign is a sign of a quality; on this point he makes the following observations:

"Considérés comme signes d'opérations arithmétiques, + et − sont les signes, l'un de l'addition, l'autre de la soustraction."

"Considérés comme signes d' opérations géométriques, ils indiquent des directions opposées. Si l'un, par exemple, signifie qu'une ligne doit être tirée de gauche à droite, l'autre signifie qu'elle doit être tirée de droite à gauche." .....

"Mis devant une quantité, *q*, ils peuvent désigner, comme je l'ai dit, deux opérations arithmétiques opposées dont cette quantité est le sujet. Devant cette même quantité, ils peuvent désigner deux qualités opposées ayant pour sujet les unités dont cette quantité est composée.

"Dans l' algèbre ordinaire, c'est à dire, dans l' algèbre considérée comme arithmétique universelle, ou l'on fait abstraction de toute espèce de qualité, les signes + et — ne peuvent avoir que la première de ces significations". . . . toutes les fois qu'on a pour résultat d'une opération une quantité précédée du signe —, il faut, pourque ce résultat ait un sens, y considérer quelque qualité. Alors l'algèbre ne doit plus être regardée simplement comme une arithmétique universelle, mais comme une langue mathématique."

He then proceeds to the sign $\sqrt{-1}$: this he considers a sign of perpendicularity; he argues that it is a mean proportional between $+1$ and -1, and therefore must be a perpendicular; he also gives another proof that it is a perpendicular; he makes a square to revolve through 90° about one of its angular points, and observes, that if the square is positive in its first situation, it will be negative after having moved through 90°; therefore if the square in its first situation be represented by $+1$, it will in its second situation be represented by -1, and its side will in the first case be represented by $+1$ or -1, and in the second by $+\sqrt{-1}$ or $-\sqrt{-1}$; but the side of the square has moved through 90°; therefore he concludes that $\sqrt{-1}$ is a sign of perpendicularity. In the above demonstration M. Bue'e applies his method of reasoning as well to areas as to lines; but as in my treatise I have confined myself to the algebraical representation of lines, I will not make any observation respecting the force of this proof. M. Bue'e afterwards proceeds to say, that though perpendicularity is properly the only quality indicated by $\sqrt{-1}$, yet $\sqrt{-1}$ may be made to signify any other quality, provided we can reason respecting that quality in the same manner as we reason respecting perpendicularity; he then gives examples illustrative of his theory: some of these examples I cannot understand; others are more clear; but in almost all there is one great defect, viz. he is obliged to introduce some arbitrary limitation into the question, in order to make the answer agree with the root of the equation: this arises from the want of a general geometrical definition of proportion or multiplication, which is necessary to render the theory complete: he also endeavours to prove that $(\sqrt{-1})^n = n\sqrt{-1}$; but this I cannot comprehend. However, notwithstanding these defects or errors, the general principles on which he reasons are good; he evidently proceeds on the principle, that whenever in the algebraic solution of any question, we arrive at imaginary quantities as answers, we must consider that the question might have been expressed in more general terms, and that the imaginary quantities are answers to the question in this extended sense. This appears to me to be the true principle, and is analogous to our usual method of reasoning, when we arrive at a negative answer in resolving a question, which, from the manner in which it is expressed, only admits of positive answers.

The Edinburgh reviewers in their article on M. BUÉE's "mémoire," state their opinion with respect to the nature of the square roots of negative quantities in these words:

"The essential character of imaginary expressions is to denote impossibility; and nothing can deprive them of this signification, nothing like a geometrical construction can be applied to them; they are indications of the impossibility of any such construction, or of any thing that can be exhibited to the senses."

As I have already answered this objection, it will not be necessary for me to make any further remarks on this point.

In considering the evidence adduced by M. BUÉE in support of his fundamental proposition, that $\sqrt{-1}$ expresses perpendicularity, the reviewers begin with giving his reasoning on that subject, viz. $\sqrt{-1}$ is a mean proportional between $+1$ and -1, and therefore a perpendicular; and they observe with respect to his arguments, that "any imaginable conclusion might have been obtained in the same manner, the third line for example, needed not have been placed at right angles to the other two, but making an angle, suppose of 120° with one, and of 60° with the other; it would still be a mean proportional between them, and its square would be therefore, according to the above method of reasoning equal to $+1 \times -1 = -1$, so that the line itself would be equal to $\sqrt{-1}$, and thus $\sqrt{-1}$ would denote not perpendicularity, or the situation in which a line makes the adjacent angles equal, but that in which it makes one of these angles double of the other; the one of these arguments is just as good as the other, and neither of them of course is of any value."

The above objection derives its force from the want of a definition of proportion in M. BUÉE's "mémoire," as is evident from what has already been proved in this paper.

I saw M. Mourey's work in December 1828, and found that his method of considering the subject is nearly the same as the method which I have adopted in my treatise: but he has in his work a proof that every equation has as many roots as it has dimensions, which I have not in mine; this proof with a very slight alteration I communicated to the Philosophical Society at Cambridge. My reason for introducing an alteration was this: the author, after having taken (in the figure which he makes use of) as many points as the given equation has dimensions, and proved that round each point there is a curve which has certain properties, and that in each curve there is a line which will satisfy the conditions of the equation, concludes that there are as many lines which will satisfy the conditions of the equation as the equation has dimensions; which conclusion does not necessarily follow from the premises; for one curve may surround two or more of the points in his figure, in which case he ought to have proved, that if any one of the curves surrounds m of the points, there will be m lines in that curve, which satisfy the conditions required, which he has not done, therefore his proof is in that part defective; consequently an alteration was necessary; and the alteration was easily made, as it is enough to prove, that an equation of n dimensions has one root, after which it may be depressed to an equation of $n - 1$ dimensions. In all other respects the proof given by M. Mourey is remarkably clear and satisfactory, and an example of the advantages which mathematicians may derive from a knowledge of the true theory of the quantities improperly called impossible or imaginary.